## The structure sheaf on Spec R (See Har II.2, Shaf V.2.2)

We now have the tools to put the remaining structure on $X = \operatorname{Spec} R$ by defining a sheaf of rings $\mathcal{O}_X$ (or just $\mathcal{O}$) on $X$ called the structure sheaf.

Our definition can be a little opaque, so we first describe the properties we want it to have:

1.) $\mathcal{O}(X) = R$

2.) For $p \in X$, the stalk $\mathcal{O}_p = R_p$ (localization at $P$).

3.) On a distinguished open set $D(f)$, $\mathcal{O}(D(f)) = R_f$, where $R_f := R[1/f]$.

Note that if $D(f) \subseteq D(g)$, then $V(f) \supseteq V(g)$, so $(f) \subseteq (g)$ (by replacing $f$ and $g$ w/ their radical).

In particular, $f = ag$, some $a$,

so $a\left(\frac{1}{f}\right) = \frac{1}{g} \in R_f$.

So we have a natural map $R_g \longrightarrow R_f$ which is the localization map.

We now give two equivalent definitions of $\mathcal{O}$:

**Def 1**: (more intuitive)

If $U \subseteq X = \operatorname{Spec} R$ open, define

$$\mathcal{O}(U) := \varprojlim_{D(f) \subseteq U} R_f \qquad \text{(the \underline{inverse limit})}$$

$$:= \left\{ (a_f) \in \prod R_f \;\middle|\; D(f) \subseteq U \text{ and } a_g \mapsto a_f \text{ whenever } D(f) \subseteq D(g) \right\}$$

Check that this is a sheaf and $\mathcal{O}(D(f)) = R_f$.

**Remark**: This construction actually generalizes to any sheaf whose sections we know on a basis. Namely, suppose $\{B_i\}$ is a basis for the topology on $X$, and $\mathcal{F}$ a sheaf on $X$. If we know $\mathcal{F}(B_i)$ and the restriction maps $\mathcal{F}(B_i) \to \mathcal{F}(B_j)$ when $B_j \subseteq B_i$, then we can uniquely(!) reconstruct $\mathcal{F}$ as follows:

$$\mathcal{F}(U) = \varprojlim_{B_i \subseteq U} \mathcal{F}(B_i) \subseteq \prod \mathcal{F}(B_i)$$

**Def 2**: (Hartshorne) For $U \subseteq \operatorname{Spec} R$ open, define

$$\mathcal{O}(U) = \left\{ s : U \to \coprod_{p \in U} R_p \;\middle|\; s(p) \in R_p, \text{ and } (*) \right\}$$

where $(*)$ = at each $p \in U$, $\exists$ nhood $\underset{p}{\overset{\psi}{V_p}} \subseteq U$, and $f, g \in R$ s.t. for each $q \in V_p$, $g \notin q$ and $s(q) = f/g$.

We will usually use def 1, but it's good to know the def Hartshorne uses too.

Suppose $R$ is an integral domain and $X = \operatorname{Spec} R$.
Set $K$ = field of fractions of $R$.

Then for each $f \in R$, $R_f \subseteq K$. Moreover, if $D(g) \subset D(f)$, then the localization map

$$R_f \to R_g$$

is just an inclusion. This allows us to give a nice description of $\mathcal{O}_X$ in this case:

**Prop:** If $X = \operatorname{Spec} R$, $R$ an integral domain, then for $U \subseteq X$ open,

$$\mathcal{O}(U) = \bigcap_{D(f) \subseteq U} R_f \subseteq K.$$

Moreover, if $Y$ is any collection of distinguished open sets whose union is $U$, we have

$$\mathcal{O}(U) = \bigcap_{D(f) \in Y} R_f \subseteq K.$$

**Pf:** Let $(a_f) \in \mathcal{O}(U) \subseteq \prod R_f$. Then for
We know that $a_f = a_{fg} = a_g$ in $K$. Thus, all coordinates are equal in $K$.

Set $a = a_f$. Then $a \in R_g \ \forall \ D(g) \subseteq U$, so

$a \in \bigcap R_f \Rightarrow \mathcal{O}(U) = \bigcap_{D(f) \subseteq U} R_f$.

For the second statement, we know $\mathcal{O}(U) \subseteq \bigcap_{D(f) \in Y} R_f$.

Suppose $a \in \bigcap_{D(f) \in Y} R_f$. Then the sheaf condition tells us there is a <u>unique</u> $(a_f) \in \mathcal{O}(U)$ s.t. $a_f = a$. $\square$

Notice that this proposition won't hold if $R$ has zero-divisors: If $f, g \neq 0$ s.t. $fg = 0$, then e.g. $\mathcal{O}(X) = \mathcal{O}(D(1)) = R \longrightarrow R_f$ is not an inclusion, since

$$g \longmapsto \frac{g}{1} = \frac{gf}{f} = 0.$$

Moreover, $R_f$ and $R_g$ aren't contained in a common ring.

<u>Prop:</u> If $P \in X = \operatorname{Spec} R$, then $\mathcal{O}_P \cong R_P$. i.e. the stalk is the ring localized at the corresponding prime ideal.

<u>Pf:</u> First we define a homomorphism $\varphi : \mathcal{O}_P \longrightarrow R_P$.

Let $s_p \in \mathcal{O}_P$ and choose a representative $(s, D(f))$ whose germ is $s_p$.

Then $s = \frac{a}{f^n}$.

$P \in D(f) \Rightarrow P \notin V(f)$, so $f \notin P$. Thus, $\frac{a}{f^n} \in R_p$, so set

$$\varphi(s_p) = s.$$

To see that this is well-defined, take $(t, D(g))$ another representative. Then again $g \notin P$, so $fg \in p$, so $t = s$ in $R_{fg}$, so their images agree in $R_p$.

For surjectivity, take $\frac{a}{f} \in R_p$. Then $f \notin P$, so $P \in D(f)$. Thus, the germ of $\left(\frac{a}{f}, D(f)\right)$ at $P$ is mapped to $\frac{a}{f}$ by $\varphi$.

For injectivity, suppose $\varphi(s) = \varphi(t)$. Then we can find a basis neighborhood $D(f)$ s.t. $\frac{a}{f^n}, \frac{b}{f^m} \in R_f$ have germs $s$ and $t$, respectively.

Thus $\frac{a}{f^n} = \frac{b}{f^m} \in R_p$, so there is some $h \notin P$ s.t.

$h(af^m - bf^n) = 0$. But then

$\frac{a}{f^n} = \frac{b}{f^m} \in R_{fh}$ so $s = t$.

Thus, $\varphi$ is an isomorphism. $\square$